

# THIRD MOMENT OF THE REMAINDER TERM IN WEYL'S LAW FOR HEISENBERG MANIFOLDS

MAHTA KHOSRAVI

**ABSTRACT.** Let  $R(t)$  be the remainder term in Weyl's law for a 3-dimensional Riemannian Heisenberg manifold with a certain 'arithmetic' metric. We prove a third moment result stating that  $\int_1^T R(t)^3 dt = d_3 T^{13/4} + O_\delta(T^{45/14+\delta})$ , where  $d_3$  is a specific positive constant which can be evaluated explicitly. This proves the asymmetric behavior of  $R(t)$  about the  $t$ -axis. This result is consistent with the conjecture of Petridis and Toth stating that  $R(t) = O_\delta(t^{3/4+\delta})$ . Similar results hold for  $2n+1$ -dimensional Heisenberg manifolds with arithmetic metrics.

## 1. INTRODUCTION

Let  $(M, g)$  be a closed  $n$ -dimensional Riemannian manifold with metric  $g$  and Laplace-Beltrami operator  $\Delta$ . We denote its spectral counting function by  $N(t)$ , defined as the number of the eigenvalues of  $\Delta$  not exceeding  $t$ . A celebrated theorem of Hörmander [Hö] asserts that as  $t \rightarrow \infty$ ,

$$(1) \quad N(t) = \frac{\text{vol}(B_n)\text{vol}(M)}{(2\pi)^n} t^{n/2} + O(t^{(n-1)/2}),$$

where  $\text{vol}(B_n)$  is the volume of the  $n$ -dimensional unit ball.

By considering the unit sphere, it is straightforward to show that the estimate for the remainder term in Hörmander's theorem defined by

$$(2) \quad R(t) = N(t) - \frac{\text{vol}(B_n)\text{vol}(M)}{(2\pi)^n} t^{n/2},$$

is in general sharp. However, the question of determining the optimal bound for this remainder term for any given manifold is a difficult one and depends on the properties of the associated geodesic flow. In many cases, this is an open problem. Nevertheless, for certain types of manifolds some improvements have been obtained and in a few cases the conjectured optimal bound has been attained (see [BG], [Bé], [Bl], [Fr], [Gö], [Hu], [Iv], [KP] and [Vo]).

The results obtained in this direction can be separated into two categories: (i) upper and lower bounds for the rate of growth of the remainder term (i.e. the  $O$ -results and  $\Omega$ -results respectively); (ii) the distribution of the remainder term about the  $t$ -axis and averages and moments of the remainder term.

In this article, we address a result of type (ii) on Heisenberg manifolds. Following our previous work in [KT] where we evaluated the second moment of the remainder

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term in Weyl's law for Heisenberg manifolds, we investigate the third moment of this remainder term.

We first review some well known results. For manifolds with completely integrable geodesic flows satisfying some clean intersection hypothesis, Duistermaat and Guillemin [DG] have proven that  $R(t) = o(t^{(n-1)/2})$ . For generic convex surfaces of revolution, Colin de Verdière [Co] showed that  $R(t) = O(t^{1/3})$ . The simplest compact manifold with integrable geodesic flow is the 2-torus  $\mathbb{T}^2$ . Hardy's conjecture for  $\mathbb{T}^2$  [Ha] asserts that

$$R(t) = O_\delta(t^{\frac{1}{4}+\delta}),$$

where here, and hereafter in this article,  $\delta$  is any arbitrary small positive number and the  $O_\delta$  notation indicates the implied constant may depend on the value of  $\delta$ . Hardy further proved that for  $\mathbb{T}^2$  this is the best possible upper bound. To be more precise, he proved the following lower bound results:

$$R(t) = \Omega_-((t \log t)^{\frac{1}{4}}) \quad \text{and} \quad R(t) = \Omega_+(t^{\frac{1}{4}}).$$

These lower bound results have since been improved and the best known result today is due to Soundarajan [So] who proved that

$$R(t) = \Omega\left((t \log t)^{1/4} (\log_2 t)^{(3/4)(2^{1/3}-1)} (\log_3 t)^{-5/8}\right),$$

where  $\log_2 t = \log \log t$  and  $\log_3 t = \log \log_2 t$ . Moving to the moment results on flat tori, there is a classical result of Cramér [Cr] which states that for  $\mathbb{T}^2$

$$\int_1^T R(t)^2 dt = c_2 T^{\frac{3}{2}} + O_\delta(T^{\frac{5}{4}+\delta}),$$

as  $T \rightarrow \infty$  where  $c_2 = \frac{1}{6\pi^3} \sum_1^\infty \frac{r(n)^2}{n^{3/2}}$  with  $r(n) = \#\{(a, b) \in \mathbb{Z}^2; n = a^2 + b^2\}$ . This result is consistent with Hardy's conjecture. Tsang [Ts] has evaluated the third and fourth moments of the remainder term of Weyl's law on flat tori proving that for some specific negative constant  $c_3$  and positive constant  $c_4$

$$\int_1^T R(t)^3 dt = c_3 T^{\frac{7}{4}} + O(T^{\frac{7}{4}-\epsilon}),$$

and

$$\int_1^T R(t)^4 dt = c_4 T^2 + O(T^{2-\epsilon}),$$

as  $T \rightarrow \infty$  for some  $\epsilon > 0$ . The fifth moment result on flat tori is due to the author [Kh] who has recently proven that

$$\int_1^T R(t)^5 dt = c_5 T^{\frac{9}{4}} + O_\delta(T^{\frac{727}{324}+\delta}),$$

as  $T \rightarrow \infty$  where  $c_5$  is a specific negative constant.

From the work of Heath-Brown [HB] in 1992, we know that the normalized remainder term  $t^{-1/4}R(t)$  has an asymptotic distribution function in the sense that for any interval  $I$

$$T^{-1} \text{mes} \left\{ t \in [1, T], t^{-\frac{1}{4}} R(t) \in I \right\} \longrightarrow \int_I f(\alpha) d\alpha$$

as  $T \rightarrow \infty$ . He showed that the density function and its derivatives decay on the real line faster than exponentially. His methods also show the convergence of the

moments up to order nine even though they are not strong enough to provide the rate of convergence.

As the first, natural, non-commutative generalization of  $\mathbb{T}^2$  consider 3-dimensional Heisenberg manifolds  $(\Gamma \backslash H_1, g)$ . These manifolds have completely integrable geodesic flows [Bu]. Petridis and Toth [PT] proved that for certain ‘arithmetic’ Heisenberg metrics  $R(t) = O_\delta(t^{5/6+\delta})$ . Later in [CPT] the exponent was improved to  $R(t) = O_\delta(t^{34/41+\delta})$  and the result extended to all left-invariant Heisenberg metrics. It was conjectured in [PT] that for  $(\Gamma \backslash H_1, g)$ ,

$$(3) \quad R(t) = O_\delta(t^{\frac{3}{4}+\delta}).$$

Moreover, as evidence for this conjecture, Petridis and Toth [PT] proved the following  $L^2$ -result for  $(\Gamma \backslash H_1, g)$  with the arithmetic metric by averaging locally over the moduli space of left-invariant metrics

$$\int_{I^3} |N(t; u) - \frac{1}{6\pi^2} \text{vol}(M(u)) t^{\frac{3}{2}}|^2 du \leq C_\delta t^{\frac{3}{2}+\delta},$$

where  $I = [1 - \epsilon, 1 + \epsilon]$ . They also proved that for sufficiently large  $T$ ,

$$\frac{1}{T} \int_T^{2T} |N(t) - \frac{1}{6\pi^2} \text{vol}(M) t^{\frac{3}{2}}| dt \gg T^{\frac{3}{4}}.$$

It has been noted that the conjecture (3) follows from the standard conjectures on the growth of exponential sums, see [CPT].

In higher dimensions, i.e.  $(\Gamma \backslash H_n, g)$  where  $n > 1$ , in joint work with Petridis [KP] we proved that for generic irrational metrics

$$R(t) = O_\delta(t^{n-\frac{1}{4}+\delta}).$$

Moreover, we demonstrated that this bound is sharp.

As evidence for (3), we proved with Toth [KT] the  $L_2$ -result

$$(4) \quad \int_1^T R(t)^2 dt = d_2 T^{\frac{5}{2}} + O_\delta(T^{\frac{9}{4}+\delta}),$$

where  $d_2$  is an explicitly evaluated positive constant.

The main purpose of this paper is to prove that  $\int_1^T R(t)^3 dt$  similarly has meaningful asymptotics for  $(\Gamma \backslash H_n, g)$ .

**Theorem 1.1.** *For  $(2n+1)$ -dimensional Heisenberg manifold with the metric  $g = \begin{pmatrix} I_{2n \times 2n} & 0 \\ 0 & 2\pi \end{pmatrix}$ , where  $I_{2n \times 2n}$  is the identity matrix, there exists a positive constant  $d_3$  such that*

$$(5) \quad \int_1^T R(t)^3 dt = d_3 T^{3n+\frac{1}{4}} + O_\delta(T^{3n+\frac{3}{14}+\delta}).$$

**Remark 1.** *Without loss of generality, we prove Theorem 1.1 only for 3-dimensional Heisenberg manifolds. In general dimensions the proof follows in an identical manner. See [KT] for the exponential representation of the remainder term of Weyl’s law on higher-dimensional Heisenberg manifolds.*

**Remark 2.** *Theorem 1.1 also holds for rational  $(2n + 1)$ -dimensional Heisenberg manifolds (for the definition of rationality refer to [KP]). However in this case, our methods to prove the positivity of the constant do not apply any more. In the case of irrational Heisenberg manifolds we do not currently know how to prove the result.*

**Remark 3.** *Based on the the method of the proof, which implies a large truncation index in the summation defining the mollified remainder term, we are not able to modify this method to prove a 4th moment result.*

## 2. BACKGROUND ON HEISENBERG MANIFOLDS

We review here some of the basic properties of Heisenberg manifolds. The reader should consult [GW], [St] or [Fo] for further details.

**2.1. Basic definitions and notation.** For any two real numbers  $x$  and  $y$  let

$$\gamma(x, y, t) = \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad X(x, y, t) = \begin{pmatrix} 0 & x & t \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

The real 3-dimensional Heisenberg group  $H_1$  is the Lie subgroup of  $Gl_3(\mathbb{R})$  consisting of all matrices of the form  $\gamma(x, y, t)$ :

$$H_1 = \{\gamma(x, y, t) : x, y \in \mathbb{R}, t \in \mathbb{R}\}.$$

The Lie algebra of  $H_1$  is:

$$\mathfrak{h}_1 = \{X(x, y, t) : x, y \in \mathbb{R}, t \in \mathbb{R}\}.$$

The matrix exponential maps  $\mathfrak{h}_1$  diffeomorphically onto  $H_1$  and is given by the formula

$$\begin{cases} \exp : \mathfrak{h}_1 \mapsto H_1, \\ X(x, y, t) \mapsto \gamma(x, y, t + \frac{1}{2}xy). \end{cases}$$

The product operation in  $H_1$  and Lie bracket in  $\mathfrak{h}_1$  are given by

$$\begin{aligned} \gamma(x, y, t) \cdot \gamma(x', y', t') &= \gamma(x + x', y + y', t + t' + xy'), \\ [X(x, y, t), X(x', y', t')] &= X(0, 0, xy' - x'y). \end{aligned}$$

The algebra  $\mathfrak{z}_1 = \{X(0, 0, t), t \in \mathbb{R}\}$  is both the center and the derived subalgebra of  $\mathfrak{h}_1$ . It is also convenient to identify the subspace  $\{X(x, y, 0), x, y \in \mathbb{R}\}$  of  $\mathfrak{h}_1$  with  $\mathbb{R}^2$  and so,  $\mathfrak{h}_1 = \mathbb{R}^2 \oplus \mathfrak{z}_1$ .

The standard basis of  $\mathfrak{h}_1$  is the set  $\delta = \{X_1, Y_1, Z\}$ , where the first 2 elements are the standard basis of  $\mathbb{R}^2$  and  $Z = X(0, 0, 1)$ . The only nonzero bracket among the elements of  $\delta$  is given by  $[X_1, Y_1] = Z$ .

**Definition 2.1.** *A Riemannian Heisenberg manifold is a pair  $(\Gamma \backslash H_1, g)$  where  $\Gamma$  is a uniform discrete subgroup of  $H_1$  ('uniform' means that the quotient  $\Gamma \backslash H_1$  is compact), and  $g$  is a Riemannian metric on  $\Gamma \backslash H_1$  whose lift to  $H_1$  is left  $H_1$ -invariant.*

**2.2. Classification of the uniform discrete subgroups of  $H_1$ .** For every positive integer  $r$ , define

$$\Gamma_r = \{\gamma(x, y, t) : x \in r\mathbb{Z}, y \in \mathbb{Z}, t \in \mathbb{Z}\}.$$

It is clear that  $\Gamma_r$  is a uniform discrete subgroup of  $H_1$ .

**Theorem 2.2.** ([GW], Theorem 2.4) *The subgroups  $\Gamma_r$  classify the uniform discrete subgroups of  $H_1$  up to automorphisms. In other words, for every uniform discrete subgroup of  $H_1$  there exists a unique  $r \in \mathbb{Z}_+$  and an automorphism of  $H_1$  which maps  $\Gamma$  to  $\Gamma_r$ . Also, if two subgroups  $\Gamma_r$  and  $\Gamma_s$  are isomorphic then  $r$  and  $s$  are equal.*

**Corollary 2.3.** ([GW], Corollary 2.5) *Given any Riemannian Heisenberg manifold  $M = (\Gamma \backslash H_1, g)$ , there exists a unique positive integer  $r$  and a left-invariant metric  $\tilde{g}$  on  $H_1$  such that  $M$  is isometric to  $(\Gamma_r \backslash H_1, \tilde{g})$ .*

Since every left-invariant metric  $g$  on  $H_1$  is uniquely determined by an inner product on  $\mathfrak{h}_1$ , the left-invariant metrics can be identified with their matrices relative to the standard basis of  $\mathfrak{h}_1$ . For any  $g$  we can choose an inner automorphism  $\varphi$  of  $H_1$  such that  $\mathbb{R}^2$  is orthogonal to  $\mathfrak{z}_1$  with respect to  $\varphi^*g$ . Therefore,  $(\Gamma \backslash H_1, g)$  will be isometric to  $(\Gamma \backslash H_1, \varphi^*g)$  and we can replace every left-invariant metric  $g$  by  $\varphi^*g$  and always assume that the metric  $g$  has the form  $g = \begin{pmatrix} h & 0 \\ 0 & g_3 \end{pmatrix}$ , where  $h$  is a positive-definite  $2 \times 2$  matrix and  $g_3$  is a positive real number. The volume of the Heisenberg manifold is given by the formula  $\text{vol}(\Gamma_r \backslash H_1, g) = r\sqrt{\det(g)}$ .

**2.3. The spectrum of Heisenberg manifolds.** Let  $M = (\Gamma \backslash H_1, g)$  be a Heisenberg manifold where the metric  $g$  is in the arithmetic form  $g = \begin{pmatrix} I_2 & 0 \\ 0 & 2\pi \end{pmatrix}$  and  $I_2$  is the two by two identity matrix.

Let  $\Sigma$  be the spectrum of the Laplacian on  $M = (\Gamma \backslash H_1, g)$ , where the eigenvalues are counted with multiplicities. Then,  $\Sigma = \Sigma_1 \cup \Sigma_2$  ( see [GW] page 258) where,

$$\Sigma_1 = \{\lambda(m, n) = 4\pi^2(m^2 + n^2); (m, n) \in \mathbb{Z}^2\},$$

such that  $\lambda(m, n)$  is counted once for each pair  $(m, n) \in \mathbb{Z}^2$  such that  $\lambda = \lambda(m, n)$ . The second part of the spectrum,  $\Sigma_2$ , is the set

$$\Sigma_2 = \{\mu(c, k) = 2\pi c(c + (2k + 1)); c \in \mathbb{Z}_+, k \in (\mathbb{Z}_+ \cup \{0\})\},$$

where every  $\mu(c, k)$  is counted with multiplicity  $2c$ .

### 3. ESTIMATES FOR REGULARIZED SPECTRAL COUNTING FUNCTION

The idea of the proof of theorem (1.1) is to use the exponential sum representation which we proved for the regularized spectral counting function in [KT] and apply a modified version of the method used by Tsang [Ts].

In this section we give a short overview on some of the notation and results proved in [KT]. Let  $N(t)$  to be the spectral counting function defined by

$$(6) \quad N(t) = N_T(t) + N_H(t),$$

where  $N_T(t)$  is the spectral counting function of the torus, defined by

$$N_T(t) = \#\{\lambda \in \Sigma_1; \lambda \leq t\},$$

and  $N_H(t)$  is defined by

$$N_H(t) = \#\{\lambda \in \Sigma_2; \lambda \leq t\}.$$

The estimates for  $N_T(t)$  are well-known. For example,

$$(7) \quad N_T(t) = \frac{t}{4\pi} + O(t^{\frac{1}{2}}),$$

will suffice for our purposes. This bound was known to Gauss. To evaluate  $N_H(t)$ , we write

$$(8) \quad N_H(t) = \sum_{c(c+(2k+1)) \leq t/2\pi} 2c.$$

Define  $A_t = \{(x, y); x > 0, y > 0, x(x + 2y + 1) \leq t\}$ . Then, we have

$$(9) \quad N_H(2\pi t) = \sum_{(c,k) \in \mathbb{Z}^2} (2c) \chi_{A_t}(c, k).$$

To obtain an exponential-sum representation for the remainder term we need to apply the Poisson summation formula to write the remainder term, corresponding to type II eigenvalues, in a form which can be estimated by the method of the stationary phase.

However, to justify the application of the Poisson summation formula for  $N_H(2\pi t)$ , we need to regularize the characteristic function  $\chi_{A_t}$ . Take  $\rho$  to be a smooth symmetric positive function on  $\mathbb{R}^2$  with  $\int_{\mathbb{R}^2} \rho(x, y) dx dy = 1$  and  $\text{supp}(\rho) \subseteq [-1, 1]^2$ . Let  $\rho_\epsilon(x, y) = \epsilon^{-2} \rho(x/\epsilon, y/\epsilon)$ , where we make an explicit choice of  $\epsilon > 0$  later on. Consider the mollified counting functions

$$(10) \quad N_H^\epsilon(t) := \sum_{(c,k) \in \mathbb{Z}^2} (2c) \chi_{A_t}(c, k) * \rho_\epsilon(c, k).$$

**Lemma 3.1.** *Let  $T$  be an arbitrarily large number and put  $\epsilon = T^{-\gamma}$  for an arbitrary fixed  $\gamma \in (0, 1]$ . Then, for  $1 < t < T$  and a constant  $c_\gamma > 2$  which depends only on  $\gamma$ , we have*

$$N_H^\epsilon(t - c_\gamma T^{1-\gamma}) \leq N_H(2\pi t) \leq N_H^\epsilon(t + c_\gamma T^{1-\gamma}).$$

*Proof.* We prove the first inequality in 3.1. The other inequality follows in the same way. Given  $A_t = \{(x, y); x > 0, y > 0, x(x + y) \leq t\}$ , let  $\partial A_t$  to be the hyperbola  $x(x + y) = t$ . If a point  $X = (x, y) \in \mathbb{Z}_+^2$  lies at a distance greater than  $\sqrt{2}\epsilon$  from  $\partial A_t$ , then  $\chi_{A_t} * \rho_\epsilon(X) = \chi_{A_t}(X)$ .

Therefore, by taking  $\Omega_1 = \{(c, k) \in \mathbb{Z}^2; \text{dist}((c, k), \partial A_{t+K\epsilon}) > \sqrt{2}\epsilon\}$ , we have,

$$\begin{aligned}
N_H^\epsilon(t + K\epsilon) &= \sum_{(c,k) \in \mathbb{Z}^2} (2c)(\chi_{A_{t+K\epsilon}} * \rho_\epsilon)(c, k) \\
&= \sum_{(c,k) \in \Omega_1} (2c)\chi_{A_{t+K\epsilon}}(c, k) + \sum_{(c,k) \in \mathbb{Z}^2 \setminus \Omega_1} (2c)(\chi_{A_{t+K\epsilon}} * \rho_\epsilon)(c, k).
\end{aligned}$$

On the other hand,

$$N_H(2\pi t) = \sum_{(c,k) \in \mathbb{Z}^2} (2c)\chi_{A_t}(c, k).$$

So, to get  $N_H^\epsilon(t + K\epsilon) \geq N_H(2\pi t)$ , it suffices to choose  $\epsilon$  and  $K$  so that  $\mathbb{Z}^2 \cap A_t \subseteq \Omega_1$ . Since the closest point of  $\mathbb{Z}^2 \cap A_t$  to  $\partial A_{t+K\epsilon}$  is  $(1, [t-1])$ , it suffices to require that

$$(11) \quad \text{dist}((1, t), (\frac{-t + \sqrt{t^2 + 4t + 4K\epsilon}}{2}, t)) > \sqrt{2}\epsilon.$$

Equation (11) is equivalent to  $4K\epsilon > 4\epsilon^2 + 4 + 4\epsilon t + 8\epsilon$ . So, it is enough to choose  $K = 2T$  and  $\epsilon = T^{-\gamma}$ . The inequality  $N_A^\epsilon(t - c_\gamma T^{1-\gamma}) \leq N_H(2\pi t)$  can be proved in the same way.  $\square$

**Remark 4.** Lemma 3.1 will help us to convert our average results on  $N_H^\epsilon(t)$  back to  $N_H(t)$ . However, for this conversion we need  $\gamma > 3/4$ .

**Remark 5.** Based on the condition  $\gamma > 3/4$ , which implies a large truncation index in the summation defining  $R_H^\epsilon(t)$ , we are not able to modify this method to prove a 4th moment result.

**Remark 6.**

- (1) Henceforth, we always assume  $\epsilon = T^{-\gamma}$  for a fixed large  $T$ , fixed  $\gamma \in (0, 1]$  and  $t \in [1, T]$ . Also we assume that  $\delta$  is an arbitrary small positive number independent of  $T$ .
- (2) By the notation  $f(x) \ll g(x)$ , we mean that there exists a positive constant  $C$  such that  $|f(x)| \leq C|g(x)|$  for every  $x$ .

**Proposition 3.2.** ([KT]) The following asymptotic expansion holds for  $N_H^\epsilon$ :

$$(12) \quad N_H^\epsilon(t) = \frac{2}{3}t^{\frac{3}{2}} - \frac{1}{2}t + R_H^\epsilon(t) + O(t^{\frac{1}{2}+\delta}),$$

where,

$$\begin{aligned}
R_H^\epsilon(t) &= \frac{t^{\frac{3}{4}}}{\pi} \sum_{\substack{0 < \nu < \mu, \\ \mu \equiv \nu \pmod{2}}} (-1)^\nu \cos(2\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4}) \mu^{-\frac{5}{4}} \nu^{-\frac{1}{4}} \widehat{\rho}_\epsilon(\frac{\mu + \nu}{2}, \nu) \\
(13) \quad &+ \frac{t^{\frac{3}{4}}}{2\pi} \sum_{0 < \nu} (-1)^\nu \cos(2\pi\sqrt{t}\nu - \frac{\pi}{4}) \nu^{-\frac{3}{2}} \widehat{\rho}_\epsilon(\nu, \nu).
\end{aligned}$$

## 4. PROOF OF THEOREM 1.1

Given the formula for the regularized counting function in Proposition 3.2, we prove Theorem 1.1 in three steps: First, we truncate the exponential sum representing  $R_H^\epsilon$  at a suitable term. Then, we apply a modified version of Tsang's method to this truncated sum. Finally, using Lemma 3.1, we eliminate the mollifier  $\rho_\epsilon$  and prove Theorem 1.1.

**Lemma 4.1.** *Let  $F_H^\epsilon(t)$  be the first summation on the right-hand side of (13), then we have*

$$F_H^\epsilon(t) = \sum_{\substack{0 < \nu < \mu; \mu\nu < T^\alpha \\ \mu \equiv \nu \pmod{2}}} (-1)^\nu t^{\frac{3}{4}} \cos(2\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4}) \mu^{-\frac{5}{4}} \nu^{-\frac{1}{4}} \widehat{\rho}_\epsilon\left(\frac{\mu + \nu}{2}, \nu\right) + O(T^{1/2}),$$

where  $\alpha$  is an arbitrary positive number lying in  $(2\gamma, 2)$ .

*Proof.* Since  $\widehat{\rho}$  is a Schwartz function, for any positive integer  $m$  we have

$$(14) \quad \widehat{\rho}_\epsilon\left(\frac{\mu + \nu}{2}, \nu\right) << \frac{1}{(\epsilon^2 \mu \nu)^m},$$

for  $\mu > \nu > 0$ . Applying  $\epsilon = T^{-\gamma}$  and letting  $\alpha > 0$  we then have

$$(15) \quad \sum_{0 < \nu < \mu; \mu\nu \geq T^\alpha} (-1)^\nu t^{\frac{3}{4}} \cos(2\pi\sqrt{t}\sqrt{\mu\nu} - \frac{\pi}{4}) \mu^{-\frac{5}{4}} \nu^{-\frac{1}{4}} \widehat{\rho}_\epsilon\left(\frac{\mu + \nu}{2}, \nu\right) \\ \ll T^{\frac{3}{4}} \sum_{k \geq T^\alpha} k^{-m-\frac{1}{4}} \sum_{\mu|k; \mu > \sqrt{k}} \mu^{-1} \leq T^{\frac{3}{4}+2\gamma m+\alpha(\frac{1}{4}-m+\delta)}.$$

Therefore, to have this tail bounded by  $T^{\frac{3}{4}}$  we shall choose  $\alpha > \frac{2\gamma m}{m-1/4-\delta}$  and since we can choose  $m$  as large as we please, this inequality holds if we assume  $\alpha > 2\gamma$ .  $\square$

To evaluate the third moment, we have

$$\begin{aligned} \int_1^T F_H^\epsilon(t)^3 dt &= \sum_{\mu_j, \nu_j} (-1)^{\nu_1+\nu_2+\nu_3} \prod_{j=1}^3 \left\{ \mu_j^{-\frac{5}{4}} \nu_j^{-\frac{1}{4}} \widehat{\rho}_\epsilon\left(\frac{\mu_j + \nu_j}{2}, \nu_j\right) \right\} \\ &\quad \times \int_1^T t^{9/4} \prod_{j=1}^3 \cos(2\pi\sqrt{t}\sqrt{\mu_j\nu_j} - \frac{\pi}{4}) dt \\ &= \frac{1}{8} \sum_{\mu_j, \nu_j} (-1)^{\nu_1+\nu_2+\nu_3} \prod_{j=1}^3 \left\{ \mu_j^{-\frac{5}{4}} \nu_j^{-\frac{1}{4}} \widehat{\rho}_\epsilon\left(\frac{\mu_j + \nu_j}{2}, \nu_j\right) \right\} \\ &\quad \times \int_1^T t^{9/4} e^{\pi i(\pm(2\sqrt{t\mu_1\nu_1}-\frac{1}{4}) \pm (2\sqrt{t\mu_2\nu_2}-\frac{1}{4}) \pm (2\sqrt{t\mu_3\nu_3}-\frac{1}{4}))} dt, \end{aligned}$$

where  $\pm$  means that we have a total of 8 terms, one for each choice of  $+$  or  $-$  sign. Next, we show that all the indices for which  $\pm(2\sqrt{t\mu_1\nu_1}-\frac{1}{4}) \pm (2\sqrt{t\mu_2\nu_2}-\frac{1}{4}) \pm (2\sqrt{t\mu_3\nu_3}-\frac{1}{4}) \neq 0$  lead to lower order terms. Without loss of generality, we continue the proof by considering the following summation



(16)

$$S := \sum_{\substack{\Delta \neq 0 \\ \mu_1 \nu_1 \geq \mu_2 \nu_2}} (-1)^{\nu_1 + \nu_2 + \nu_3} \prod_{j=1}^3 \left\{ \mu_j^{-\frac{5}{4}} \nu_j^{-\frac{1}{4}} \widehat{\rho}_\epsilon \left( \frac{\mu_j + \nu_j}{2}, \nu_j \right) \right\} \int_1^T t^{9/4} e^{2\pi i \sqrt{t} \Delta - \frac{\pi i}{4}} dt,$$

where  $\Delta := \sqrt{\mu_1 \nu_1} + \sqrt{\mu_2 \nu_2} - \sqrt{\mu_3 \nu_3}$ . Since  $|\widehat{\rho}_\epsilon|$  is bounded above by 1, we have

$$|S| \leq \sum_{\substack{\Delta \neq 0 \\ \mu_1 \nu_1 \geq \mu_2 \nu_2}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} \left| \int_1^T t^{9/4} e^{2\pi i \sqrt{t} \Delta - \frac{\pi i}{4}} dt \right|.$$

For fixed positive  $\sigma$  and  $\beta$  to be specified later, break the summation in (17) in three parts.

*Case 1:* If  $|\Delta| > (\mu_1 \nu_1)^{1/2-\sigma}$  then using the integral estimate

$$\left| \int_1^T t^{9/4} e^{i\omega \sqrt{t}} dt \right| \leq 2 \left[ \frac{t^{11/4}}{|\omega|} \right]_1^T \ll \frac{T^{11/4}}{|\omega|},$$

we have

$$\begin{aligned} S_1 &:= \sum_{\substack{|\Delta| > (\mu_1 \nu_1)^{1/2-\sigma} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} \left| \int_1^T t^{9/4} e^{2\pi i \sqrt{t} \Delta - \frac{\pi i}{4}} dt \right| \\ &\ll T^{\frac{11}{4}} \sum_{\substack{|\Delta| > (\mu_1 \nu_1)^{1/2-\sigma} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} |\Delta|^{-1} \end{aligned}$$

Applying the condition on  $\Delta$  we find

$$\begin{aligned} S_1 &\ll T^{\frac{11}{4}} \sum_{\substack{\mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} (\mu_1 \nu_1)^{-1/2+\sigma} \\ &= T^{\frac{11}{4}} \sum_{\substack{\mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} (\mu_1 \nu_1)^{-3/4+\sigma} \mu_1^{-1} (\mu_2 \nu_2)^{-1/4} \mu_2^{-1} (\mu_3 \nu_3)^{-1/4} \mu_3^{-1} \\ (17) \quad &\ll T^{\frac{11}{4}} \sum_{\substack{\mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} (\mu_1 \nu_1)^{-1/2+\sigma+\delta} \mu_1^{-1} ((\mu_3 \nu_3)^{-1/4} \mu_3^{-1} \end{aligned}$$

$$(18) \quad \ll T^{\frac{11}{4}+\delta} \sum_{0 < m_j < T^\alpha} m_1^{-1+\sigma} m_3^{-3/4}.$$

To obtain (17) and (18) we have used the fact that

$$\sum_{\mu_j > \nu_j > 0} (\mu_j \nu_j)^{-1/4} \mu_j^{-1} \ll \sum_{m_j} m_j^{-3/4+\delta}$$

for  $m_j := \mu_j \nu_j$  and arbitrary  $\delta > 0$ .

Therefore,

$$(19) \quad S_1 = O\left(T^{11/4+\alpha/4+\delta+\sigma\alpha}\right),$$

and to have  $S_1 = o(T^{13/4})$  we need the condition

$$(20) \quad \alpha\left(\frac{1}{4} + \sigma\right) < \frac{1}{2}$$

on  $\alpha$  and  $\sigma$ .

*Case 2:* If  $|\Delta| \leq (\mu_1 \nu_1)^{1/2-\sigma}$ , then we can prove that  $\mu_3 \nu_3$  has basically the same order of magnitude as  $\mu_1 \nu_1$  and the number of the solutions for  $\mu_3 \nu_3$  satisfying  $|\Delta| \leq (\mu_1 \nu_1)^{1/2-\sigma}$  is bounded by  $1 + 5|\Delta|\sqrt{\mu_1 \nu_1}$ . To prove these claims we note that

$$\mu_3 \nu_3 = (\sqrt{\mu_1 \nu_1} + \sqrt{\mu_2 \nu_2})^2 + \Delta^2 \pm 2\Delta(\sqrt{\mu_1 \nu_1} + \sqrt{\mu_2 \nu_2}).$$

Therefore,

$$(21) \quad \left| \mu_3 \nu_3 - (\sqrt{\mu_1 \nu_1} + \sqrt{\mu_2 \nu_2})^2 \right| \leq \Delta^2 + 4|\Delta|\sqrt{\mu_1 \nu_1} \leq 5(\mu_1 \nu_1)^{1-\sigma},$$

which shows the first claim is true

$$(22) \quad \frac{\mu_1 \nu_1}{2} \leq \mu_3 \nu_3 \leq 9\mu_1 \nu_1.$$

The second claim is also clear by looking at (21) which can be written as

$$\left| \mu_3 \nu_3 - (\sqrt{\mu_1 \nu_1} + \sqrt{\mu_2 \nu_2})^2 \right| \leq 5|\Delta|\sqrt{\mu_1 \nu_1}.$$

To consider the case when  $|\Delta| \leq (\mu_1 \nu_1)^{1/2-\sigma}$ , we divide it to two subcases: the first one is if  $\Delta$  is permitted to be very small, i.e.  $|\Delta| \leq T^{-\beta}$ . The second case is if  $T^{-\beta} < |\Delta| \leq (\mu_1 \nu_1)^{1/2-\sigma}$ .

*Subcase 2.1:* If  $|\Delta| \leq T^{-\beta}$ , then by using the trivial bound on the integral we find

$$\begin{aligned} S_2 &:= \sum_{\substack{0 < |\Delta| \leq T^{-\beta} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} \left| \int_1^T t^{9/4} e^{2\pi i \sqrt{t} \Delta - \frac{\pi i}{4}} dt \right| \\ &\ll T^{\frac{13}{4}} \sum_{\substack{0 < |\Delta| \leq T^{-\beta} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}}. \end{aligned}$$

Using the fact that  $\mu_3 \nu_3$  has basically the same order of magnitude as  $\mu_1 \nu_1$  and the number of the solutions for  $\mu_3 \nu_3$  satisfying  $|\Delta| \leq (\mu_1 \nu_1)^{1/2-\sigma}$  is bounded by  $1 + 5|\Delta|\sqrt{\mu_1 \nu_1}$  we get that

$$\begin{aligned}
S_2 &\ll T^{\frac{13}{4}} \sum_{\substack{0 < |\Delta| \leq T^{-\beta} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2 \\ \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \left( \frac{\mu_1 \nu_1}{2} \right)^{-\frac{3}{4} + \delta} (1 + 5|\Delta| \sqrt{\mu_1 \nu_1}) \\
&\ll T^{\frac{13}{4}} \sum_{\substack{0 < |\Delta| \leq T^{-\beta} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2 \\ \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} (\mu_1 \nu_1)^{-1 + \delta} \mu_1^{-1} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} + 5T^{\frac{13}{4} - \beta} \sum_{\substack{\mu_1 \nu_1 \geq \mu_2 \nu_2 \\ \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} (\mu_1 \nu_1)^{-\frac{1}{2} + \delta} \mu_1^{-1} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \\
&\ll T^{\frac{13}{4}} \sum_{\substack{\mu_1 > \nu_1 > 0 \\ T^{2\beta/3} \leq \mu_1 \nu_1 < T^\alpha}} (\mu_1 \nu_1)^{-\frac{3}{4} + \delta} \mu_1^{-1} + 5T^{\frac{13}{4} - \beta} \sum_{\substack{\mu_1 > \nu_1 > 0 \\ \mu_1 \nu_1 < T^\alpha}} (\mu_1 \nu_1)^{-\frac{1}{4} + \delta} \mu_1^{-1} \\
&\ll T^{\frac{13}{4}} \sum_{T^{2\beta/3} \leq m_1} m_1^{-\frac{5}{4} + \delta} + 5T^{\frac{13}{4} - \beta} \sum_{0 < m_1 < T^\alpha} m_1^{-\frac{3}{4} + \delta}.
\end{aligned}$$

Therefore,

$$(23) \quad S_2 = O\left(T^{\frac{13}{4} - \frac{\beta}{6} + \delta} + T^{\frac{13}{4} - \beta + \frac{\alpha}{4} + \delta}\right),$$

and to have  $S_2 = o(T^{\frac{13}{4}})$  we need the condition

$$(24) \quad \frac{\alpha}{4} - \beta < 0.$$

on  $\alpha$  and  $\beta$  to be satisfied.

*Subcase 2.2:* Finally let us consider the last case, that is when  $T^{-\beta} < |\Delta| \leq (\mu_1 \nu_1)^{1/2 - \sigma}$ . We have

$$\begin{aligned}
S_3 &:= \sum_{\substack{T^{-\beta} < |\Delta| \leq (\mu_1 \nu_1)^{1/2 - \sigma} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} \left| \int_1^T t^{9/4} e^{2\pi i \sqrt{t} \Delta - \frac{\pi i}{4}} dt \right| \\
&\ll T^{\frac{11}{4}} \sum_{\substack{T^{-\beta} < |\Delta| \leq (\mu_1 \nu_1)^{1/2 - \sigma} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} |\Delta|^{-1}.
\end{aligned}$$

Like before, we use the fact that  $\mu_3 \nu_3$  has basically the same order of magnitude as  $\mu_1 \nu_1$  and the number of the solutions for  $\mu_3 \nu_3$  satisfying  $|\Delta| \leq (\mu_1 \nu_1)^{1/2 - \sigma}$  is bounded by  $1 + 5|\Delta| \sqrt{\mu_1 \nu_1}$  to write

$$\begin{aligned}
S_3 &\ll T^{\frac{11}{4}} \sum_{\substack{T^{-\beta} < |\Delta| \leq (\mu_1 \nu_1)^{1/2-\sigma} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \left( \frac{\mu_1 \nu_1}{2} \right)^{-\frac{3}{4}+\delta} (1+5|\Delta| \sqrt{\mu_1 \nu_1}) |\Delta|^{-1} \\
&\ll T^{\frac{11}{4}} \sum_{\substack{T^{-\beta} < |\Delta| \leq (\mu_1 \nu_1)^{1/2-\sigma} \\ \mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} (\mu_1 \nu_1)^{-1+\delta} \mu_1^{-1} (\mu_2 \nu_2)^{-1/4} \mu_2^{-1} |\Delta|^{-1} \\
&\quad + T^{\frac{11}{4}} \sum_{\substack{\mu_1 \nu_1 \geq \mu_2 \nu_2, \mu_j > \nu_j > 0 \\ \mu_j \nu_j < T^\alpha}} (\mu_1 \nu_1)^{-1/2+\delta} \mu_1^{-1} (\mu_2 \nu_2)^{-1/4} \mu_2^{-1} \\
&\ll T^{\frac{11}{4}+\beta} \sum_{\substack{\mu_1 > \nu_1 > 0 \\ \mu_1 \nu_1 < T^\alpha}} (\mu_1 \nu_1)^{-3/4+\delta} \mu_1^{-1} + T^{\frac{11}{4}} \sum_{\substack{\mu_1 > \nu_1 > 0 \\ \mu_1 \nu_1 < T^\alpha}} (\mu_1 \nu_1)^{-1/4+\delta} \mu_1^{-1}.
\end{aligned}$$

Therefore

$$(25) \quad S_3 = O\left(T^{\frac{11}{4}+\beta+\delta} + T^{11/4+\alpha/4+\delta}\right),$$

and to have  $S_3 = o(T^{\frac{13}{4}})$  we require

$$(26) \quad \alpha < 2 \quad \text{and} \quad \beta < \frac{1}{2}.$$

Therefore, taking all the conditions from Remark 4, Lemma 4.1, (20), (24) and (26) together, we have proved that by making an arbitrary choice for  $\alpha$ ,  $\beta$  and  $\sigma$  satisfying

$$(27) \quad \frac{3}{2} < \alpha < 4\beta < 2 \quad \text{and} \quad 0 < \sigma < \frac{1}{2\alpha} - \frac{1}{4},$$

we have

$$\begin{aligned}
\int_1^T F_H^\epsilon(t)^3 dt &= \frac{3}{8} \sum_{\Delta=0} (-1)^{\nu_1+\nu_2+\nu_3} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} \int_1^T 2t^{9/4} \cos\left(\frac{\pi}{4}\right) dt \\
&\quad \hat{\rho}_\epsilon\left(\frac{\mu_1+\nu_1}{2}, \nu_1\right) \hat{\rho}_\epsilon\left(\frac{\mu_2+\nu_2}{2}, \nu_2\right) \hat{\rho}_\epsilon\left(\frac{\mu_3+\nu_3}{2}, \nu_3\right) + O(|S|) \\
(28) \quad &= \frac{3\sqrt{2}}{26} T^{13/4} \sum_{\Delta=0} (-1)^{\nu_1+\nu_2+\nu_3} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} \\
&\quad \hat{\rho}_\epsilon\left(\frac{\mu_1+\nu_1}{2}, \nu_1\right) \hat{\rho}_\epsilon\left(\frac{\mu_2+\nu_2}{2}, \nu_2\right) \hat{\rho}_\epsilon\left(\frac{\mu_3+\nu_3}{2}, \nu_3\right) + O(|S|),
\end{aligned}$$

where  $O(|S|) = O(S_1 + S_2 + S_3) = o(T^{13/4})$ . Next we split the summation in (28) into the pieces where  $\mu_3 < T^{1/4}$  and  $\mu_3 \geq T^{1/4}$ . We claim that the piece where  $\mu_3 \geq T^{1/4}$  is residual. To see this, note that if  $\sqrt{\mu_1 \nu_1} + \sqrt{\mu_2 \nu_2} = \sqrt{\mu_3 \nu_3}$  then there exists integers  $k$ ,  $m_1$ ,  $m_2$  and  $m_3$  such that  $\mu_j \nu_j = k m_j^2$  and  $m_1 + m_2 = m_3$ .

Therefore,

$$\begin{aligned}
T^{\frac{13}{4}} \sum_{\substack{0 < \nu_j < \mu_j; \\ \nu_j \equiv \mu_j \pmod{2} \\ \Delta = 0; \mu_3 \geq T^{1/4}}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} &\leq T^3 \sum_{\substack{0 < \nu_j < \mu_j \\ \Delta = 0}} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{1}{4}} \nu_3^{-\frac{1}{4}} \\
&\leq T^3 \sum_{m_1 > 0; m_2 > 0; k > 0} k^{-\frac{3}{4}} m_1^{-\frac{1}{2}} m_2^{-\frac{1}{2}} (m_1 + m_2)^{-\frac{1}{2}} \sum_{\mu_j | km_j^2; \mu_j > k^{1/2} m_j} \mu_1^{-1} \mu_2^{-1} \\
(29) \quad &\leq T^3 \sum_{m_1 > 0; m_2 > 0; k > 0} k^{-\frac{7}{4}} m_1^{-\frac{3}{2}} m_2^{-\frac{3}{2}} (m_1 + m_2)^{-\frac{1}{2}} d(km_1^2) d(km_2^2) \ll T^3.
\end{aligned}$$

On the other hand, if  $\mu_3 < T^{1/4}$  then for  $j = 1, 2$  we have  $\mu_j \leq \mu_j \nu_j \leq \mu_3 \nu_3 \leq T^{1/2}$ . Since  $\epsilon = T^{-\gamma} \ll T^{-3/4}$ , we have that  $\epsilon \nu_j < \epsilon \mu_j < T^{-1/4}$  for  $j = 1, 2, 3$ . Therefore, we expand the functions  $\hat{\rho}_\epsilon(\frac{\mu_j + \nu_j}{2}, \nu_j)$  in Taylor series around the point  $(0, 0)$  and use (29), so that we can evaluate the summation in (28) as

$$\int_1^T F_H^\epsilon(t)^3 dt = \frac{3\sqrt{2}}{26} T^{\frac{13}{4}} \sum_{\substack{0 < \nu_j < \mu_j; \\ \nu_j \equiv \mu_j \pmod{2} \\ \Delta = 0}} (-1)^{\nu_1 + \nu_2 + \nu_3} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} + O(|S|).$$

Repeating a similar argument for the second summation in  $R_H^\epsilon(t)$  given in (13), we have proved that

$$(30) \quad \int_1^T R_H^\epsilon(t)^3 dt = b_3 T^{13/4} + O(|S|),$$

where  $b_3$  is the constant defined by

$$\begin{aligned}
b_3 &= \frac{3\sqrt{2}}{26\pi^3} \sum_{\substack{0 < \nu_j < \mu_j; \nu_j \equiv \mu_j \pmod{2} \\ \sqrt{\mu_1 \nu_1} + \sqrt{\mu_2 \nu_2} = \sqrt{\mu_3 \nu_3}}} (-1)^{\nu_1 + \nu_2 + \nu_3} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}} \\
&+ \frac{3\sqrt{2}}{208\pi^3} \sum_{\substack{0 < \nu_j \\ \nu_1 + \nu_2 = \nu_3}} (-1)^{\nu_1 + \nu_2 + \nu_3} \nu_1^{-\frac{3}{2}} \nu_2^{-\frac{3}{2}} \nu_3^{-\frac{3}{2}} \\
&+ \frac{9\sqrt{2}}{52\pi^3} \sum_{\substack{0 < \nu_j < \mu_j; \nu_j \equiv \mu_j \pmod{2} \\ \sqrt{\mu_1 \nu_1} + \sqrt{\mu_2 \nu_2} = \nu_3}} (-1)^{\nu_1 + \nu_2 + \nu_3} \mu_1^{-\frac{5}{4}} \nu_1^{-\frac{1}{4}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \nu_3^{-\frac{3}{2}} \\
(31) \quad &+ \frac{9\sqrt{2}}{104\pi^3} \sum_{\substack{0 < \nu_j < \mu_j; \nu_j \equiv \mu_j \pmod{2} \\ \sqrt{\mu_3 \nu_3} = \nu_1 + \nu_2}} (-1)^{\nu_1 + \nu_2 + \nu_3} \nu_1^{-\frac{3}{2}} \mu_2^{-\frac{5}{4}} \nu_2^{-\frac{1}{4}} \mu_3^{-\frac{5}{4}} \nu_3^{-\frac{1}{4}}.
\end{aligned}$$

Now to prove that  $b_3$  is a positive constant, we show that every summation on the right hand side of (31) is positive. Since every one of these summations is a special case of the first sum where one impose an extra condition that  $\mu_j = \nu_j$  for one or two of  $j$ s, we may restrict our attention only at the first sum. From  $\sqrt{\mu_1 \nu_1} = \sqrt{\mu_2 \nu_2} + \sqrt{\mu_3 \nu_3}$ , we get that for some square free  $k$  and integers  $m_1, m_2$  and  $m_3$  satisfying  $m_1 + m_2 = m_3$ , we have  $\mu_j \nu_j = m_j^2 k$ . From the conditions that  $\nu_j \equiv \mu_j$

(mod 2) and also that  $k$  is square free, we can see that  $\nu_j$  and  $m_j$  should have the same parity. Therefore from  $m_1 + m_2 = m_3$ , we get  $(-1)^{\nu_1 + \nu_2 + \nu_3} = 1$ . This proves that the first series is positive and similarly the other three are also positive. So the constant  $b_3$  is strictly positive.

The last step in the proof of the Theorem 1.1 is to use Lemma 3.1 to get rid of the mollification in  $\rho_\epsilon$  and prove the third moment estimate for  $R_H(t)$ , which is the remainder term corresponding to type II eigenvalues. From Lemma 3.1 by choosing  $\epsilon = T^{-\gamma}$  and  $1 < t < T$  for  $\gamma > 3/4$  we find

$$(32) \quad N_H^\epsilon(t - c_\gamma T^{1-\gamma}) \leq N_H(2\pi t) \leq N_H^\epsilon(t + c_\gamma T^{1-\gamma}).$$

From Proposition 3.2 we have

$$(33) \quad N_H^\epsilon(t \pm c_\gamma T^{1-\gamma}) = \frac{2}{3}t^{\frac{3}{2}} - \frac{1}{2}t + R_H^\epsilon(t \pm c_\gamma T^{1-\gamma}) + O(T^{\frac{3}{2}-\gamma}).$$

Therefore, from (32) and (33) we find

$$(34) \quad R_H^\epsilon(t - c_\gamma T^{1-\gamma}) + O(T^{\frac{3}{2}-\gamma}) \leq R_H(2\pi t) \leq R_H^\epsilon(t + c_\gamma T^{1-\gamma}) + O(T^{\frac{3}{2}-\gamma}),$$

where

$$(35) \quad R_H(2\pi t) := N_H(2\pi t) - \frac{2}{3}t^{\frac{3}{2}} + \frac{1}{2}t.$$

Taking the third moment of the right hand side of (34), we obtain

$$(36) \quad \begin{aligned} \int_1^T R_H(2\pi t)^3 dt &\leq \int_1^T R_H^\epsilon(t + c_\gamma T^{1-\gamma})^3 dt + O(T^{\frac{3}{2}-\gamma}) \int_1^T R_H^\epsilon(t + c_\gamma T^{1-\gamma})^2 dt \\ &\quad + O(T^{3-2\gamma}) \int_1^T R_H^\epsilon(t + c_\gamma T^{1-\gamma}) dt + O(T^{\frac{11}{2}-3\gamma}). \end{aligned}$$

Next we use the second moment result proved in [KT] which states that

$$(37) \quad \int_1^T R_H^\epsilon(t)^2 dt = d_2 T^{\frac{5}{2}} + o(T^{\frac{5}{2}}),$$

and, applying a simple Hölder inequality, shows that

$$(38) \quad \int_1^T |R_H^\epsilon(t)| dt = O(T^{\frac{7}{4}}).$$

Applying the results from (37) and (38) back in (36) proves that

$$\begin{aligned} \int_1^T R_H(2\pi t)^3 dt &\leq \int_1^T R_H^\epsilon(t + c_\gamma T^{1-\gamma})^3 dt + O(T^{4-\gamma}) + O(T^{\frac{19}{4}-2\gamma}) + O(T^{\frac{11}{2}-3\gamma}) \\ &\leq \int_{1+c_\gamma T^{1-\gamma}}^{T+c_\gamma T^{1-\gamma}} R_H^\epsilon(t)^3 dt + O(T^{4-\gamma}). \end{aligned}$$

Finally, we use the result from (30) and prove that

$$(39) \quad \int_1^T R_H(2\pi t)^3 dt \leq b_3 T^{\frac{13}{4}} + O(|S|) + O(T^{4-\gamma}).$$

Similarly, from the first inequality in (34) we find

$$(40) \quad \int_1^T R_H(2\pi t)^3 dt \geq b_3 T^{\frac{13}{4}} + O(|S|) + O(T^{4-\gamma}).$$

From (39) and (40) and using  $|S| \leq S_1 + S_2 + S_3$  we have

$$(41) \quad \int_1^T R_H(2\pi t)^3 dt = b_3 T^{\frac{13}{4}} + O(S_1) + O(S_2) + O(S_3) + O(T^{4-\gamma}).$$

Solving an optimization problem on the parameters  $\gamma$ ,  $\alpha$ ,  $\beta$  and  $\sigma$  satisfying the conditions

$$(42) \quad \frac{3}{2} < 2\gamma < \alpha < 4\beta < 2 \quad \text{and} \quad 0 < \sigma < \frac{1}{2\alpha} - \frac{1}{4},$$

we find that for an arbitrary small positive  $\delta' < 1/7$  and  $\gamma = 11/14$ ,  $\alpha = 11/7 + \delta'$ ,  $\beta = 3/7$  and  $\sigma = \delta'/4$ , we obtain

$$(43) \quad \int_1^T R_H(2\pi t)^3 dt = b_3 T^{\frac{13}{4}} + O_\delta(T^{\frac{13}{4} - \frac{1}{28} + \delta}),$$

for any arbitrary small positive  $\delta$ . This shows that

$$(44) \quad \int_1^T R(t)^3 dt = d_3 T^{\frac{13}{4}} + O_\delta(T^{\frac{13}{4} - \frac{1}{28} + \delta}),$$

where  $d_3 = (2\pi)^{-9/4} b_3$ . This completes the proof of Theorem 1.1.

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SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NJ 08540, USA  
*E-mail address:* `khosravi@math.ias.edu`; *Current E-mail address:* `khosravi@math.jhu.edu`